



Logarithmic Harnack inequalities for homogeneous graphs



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ABSTRACT

In this paper, we prove logarithmic Harnack inequalities for homogeneous graphs. As a consequence, we derive lower estimates for the log-Sobolev constant, extending previous results for Ricci flat graphs.

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1. Introduction

Suppose G is a graph with vertex set V and edge set E . The degree of vertex x , denoted by d_x , is the number of edges connected to x . If for every vertex x of V , d_x is finite, we say that G is a locally finite graph. The distance between two vertices is the minimum number of edges to connect them, while the diameter of G is the maximum of all the distances of the graph. We denote $x \sim y$ if vertex x is adjacent to vertex y , and μ_{xy} is the edge weight. Moreover, suppose a group χ acts on V such that:

- (i) for all $a \in \chi$, $\{au, av\} \in E$ if and only if $\{u, v\} \in E$;
- (ii) for any two vertices u and v , there is an $a \in \chi$ such that $au = v$.

Then we say G is a homogeneous graph with the associated group χ . Furthermore, we describe the edge set by an edge generating set $K \subset \chi$, then, for some $v \in V$ and $a \in K$, each edge of G is of the form $\{v, av\}$. We let K consist of k generators and require K to be symmetric, i.e., $a \in K$ if and only if $a^{-1} \in K$. If for every element $a \in K$, we have $aKa^{-1} = K$, we say that a homogeneous graph is invariant. If χ is abelian, we say G is an abelian homogeneous graph.

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Let $V^R = \{f \mid f : V \rightarrow R\}$, and the Laplace operator $\mathcal{L}$ of a graph G be

$$\mathcal{L}f(x) = \frac{1}{k} \sum_{a \in K} [f(x) - f(ax)], \quad \forall f \in V^R.$$

Suppose a function $f : V \rightarrow R$ satisfies $\mathcal{L}f(x) = \lambda f(x)$, then f is called an eigenfunction of Laplace operator $\mathcal{L}$ on graph G with eigenvalue λ , and we can easily note that 0 is a trivial eigenvalue of $\mathcal{L}$ associated with the constant eigenfunction.

According to Bakry and Emery [1], we can define a bilinear operator $\Gamma : V^R \times V^R \rightarrow V^R$ by

$$\Gamma(f, g)(x) = \frac{1}{2} \{f(x)\mathcal{L}g(x) + g(x)\mathcal{L}f(x) - \mathcal{L}(f(x)g(x))\},$$

and then the Ricci curvature operator on graphs Γ_2 by iterating Γ as

$$\Gamma_2(f, g)(x) = \frac{1}{2} \{\Gamma(f, \mathcal{L}g)(x) + \Gamma(g, \mathcal{L}f)(x) - \mathcal{L}\Gamma(f, g)(x)\}.$$

More explicitly, we have

$$\begin{aligned} \rho(x) &= \frac{1}{k} \sum_{a \in K} [f(x) - f(ax)]^2, \\ \Gamma(f, f)(x) &= \frac{1}{2} \rho(x) = \frac{1}{2} \cdot \frac{1}{k} \sum_{a \in K} [f(x) - f(ax)]^2. \end{aligned}$$

Definition 1.1. The operator $\mathcal{L}$ satisfies the curvature-dimension type inequality $CD(m, \xi)$ for some $m > 1$ if for every $f \in V^R$,

$$\Gamma_2(f, f)(x) \geq \frac{1}{m} (\mathcal{L}f(x))^2 + \xi \Gamma(f, f)(x).$$

We call m the dimension of the operator $\mathcal{L}$ and ξ the lower bound of the Ricci curvature of the operator $\mathcal{L}$. It is easy to see that for $m < \tilde{m}$, the operator $\mathcal{L}$ satisfies $CD(\tilde{m}, \xi)$ if it satisfies $CD(m, \xi)$.

If $\Gamma_2(f, f)(x) \geq \xi \Gamma(f, f)(x)$, we say that $\mathcal{L}$ satisfies $CD(\infty, \xi)$.

In 1994, F. Chung and S.T. Yau in [4] established the following Harnack inequality as in [2] for homogeneous graphs and subgraphs G with edge generating set K consisting of k generators,

$$\frac{1}{k} \sum_{a \in K} [f(x) - f(ax)]^2 + \alpha \lambda f^2(x) \leq \frac{\lambda \alpha^2}{\alpha - 2} \sup_{y \in K} f^2(y)$$

for any $\alpha > 2$ and $x \in V$, and using this Harnack inequality, they derived lower bounds for the Neumann eigenvalues and the Dirichlet eigenvalues in [4] and [6] respectively.

In 1996, F.R.K. Chung and S.T. Yau in [5] proved the logarithmic Harnack inequality for Ricci flat graphs. In fact, a homogeneous graph associated with an abelian group is Ricci flat such as in [5]. Furthermore, they derived a lower bound for the log-Sobolev constant of Ricci flat graphs using the logarithmic Harnack inequality.

In 2010, Y. Lin and S.T. Yau introduced in [7] the curvature-dimension type inequality $CD(m, \kappa)$ and proved that any locally finite connected graph satisfies either $CD(2, \frac{2}{d} - 1)$ if d is finite, or $CD(2, -1)$ if d is infinite, where $d = \sup_{x \in V} \sup_{y \sim x} \frac{d_x}{\mu_{xy}}$. They also proved that the Ricci flat graphs have the non-negative Ricci curvature in the sense of Bakry and Emery. In fact, in most cases, the Ricci curvature is zero, except

for some very special examples like complete graphs (every two vertices of the graph connected by an edge), the Ricci curvature is positive.

In 2014, with the assumption that the graph satisfies the curvature-dimension type inequality $CD(m, \kappa)$, F. Chung, Y. Lin and S.T. Yau in [3] established a Harnack inequality for general graphs with non-negative Ricci curvature, and they derived an eigenvalue lower bound, extending previous results for Ricci flat graphs.

In this paper, we consider logarithmic Harnack inequalities for general homogeneous graphs. As in [5], we define the log-Sobolev constant β of G as

$$\beta = \inf_{f \neq 0} \frac{\sum_{y \sim x} (f(x) - f(y))^2}{\sum_x f^2(x) d_x \log f^2(x)},$$

where functions f satisfy $\sum_x f^2(x) d_x = \sum_x d_x = \text{vol } G$. Then, from Theorem 5 in [5], we get the following lemma for homogeneous graphs.

Lemma 1.2. *For a homogeneous graph G , suppose $f : V \rightarrow \mathbb{R}$ achieves the log-Sobolev constant and $\sum_x f^2(x) d_x = \text{vol } G$. Then f satisfies, for any vertex x , $\mathcal{L}f(x) = \beta f(x) \log f^2(x)$.*

In the setting of homogeneous graphs, we establish a new formula for the Ricci curvature operator on graph G . Furthermore, assuming the curvature-dimension type inequality is satisfied, we get an improved logarithmic Harnack inequality and derive a lower bound for the log-Sobolev constant. Our main theorems are as follows.

Theorem 1.3. *Suppose G is a homogeneous graph satisfying the curvature-dimension type inequality $CD(m, \xi)$, and $f \in V^{\mathbb{R}}$ satisfies $\mathcal{L}f(x) = \beta f(x) \log f^2(x)$. Then, for $U = \sup_x |f(x)| \geq 1$, and $\sigma \geq 2$, the following inequality holds for $x \in V(G)$*

$$\sum_{a \in K} [f(x) - f(ax)]^2 \leq \frac{2k\sigma\beta(\sigma - 1 + s)}{\sigma - 2 + s} \cdot \max \left\{ U^2 \log U^2 \left[1 + \left(1 - \frac{2}{m\sigma(\sigma - 1 + s)} \right) \log U^2 \right], 1 \right\},$$

where $s > 0$ is a constant satisfying $\xi = s\beta$.

Taking $\sigma = 2$ in Theorem 1.3, then by Theorem 1.3 and Theorem 2.4 in Section 2, we get the following theorem. The proof is very similar to Theorem 7 in [5], so we omit the proof here.

Theorem 1.4. *Suppose G is a homogeneous graph satisfying the curvature-dimension type inequality $CD(m, \xi)$, and $f \in V^{\mathbb{R}}$ satisfies $\mathcal{L}f(x) = \beta f(x) \log f^2(x)$. Then, for $U = \sup_x |f(x)| \geq 1$, the log-Sobolev constant β of G satisfies*

$$\beta \geq \min \left(\frac{1 + 4\xi k D^2}{8k(4 - \frac{1}{m})D^2}, \frac{1}{8D^2 k(1 + \frac{1}{s}) \log U^2(1 + \log U^2)} \right),$$

where D is the diameter of G .

2. Preliminary results

Now, we establish several basic facts for homogeneous graphs. By using a modification of the proof of Theorem 1.2 in [7], we can get the following Lemma 2.1.

Lemma 2.1. *Let G be a homogeneous graph with edge generating set K consisting of k generators, then the following formula holds for the Ricci curvature operator on graph G for all $f \in V^{\mathbb{R}}$ and $x \in V$*

$$\begin{aligned}\Gamma_2(f, f)(x) &= \frac{1}{4k^2} \sum_{b \in K} \sum_{a \in K} [f(x) - f(ax) - f(bx) + f(abx)]^2 \\ &\quad - \frac{1}{2k^2} \sum_{a \in K} \left[\sum_{b \in K} (f(abx) - f(bax)) \right] [f(x) - f(ax)].\end{aligned}$$

Proof. We set

$$\rho(x) = \frac{1}{k} \sum_{a \in K} [f(x) - f(ax)]^2.$$

Then, we get

$$\begin{aligned}2\mathcal{L}\Gamma(f, f) &= \mathcal{L}\rho(x) = \frac{1}{k} \sum_{b \in K} [\rho(x) - \rho(bx)] \\ &= \frac{1}{k^2} \sum_{b \in K} \sum_{a \in K} \{ [f(x) - f(ax)]^2 - [f(bx) - f(abx)]^2 \} \\ &= -\frac{1}{k^2} \sum_{b \in K} \sum_{a \in K} [f(x) - f(ax) - f(bx) + f(abx)]^2 \\ &\quad + \frac{2}{k^2} \sum_{b \in K} \sum_{a \in K} [f(x) - f(ax) - f(bx) + f(abx)] [f(x) - f(ax)] \\ &= -\frac{1}{k^2} \sum_{b \in K} \sum_{a \in K} [f(x) - f(ax) - f(bx) + f(abx)]^2 \\ &\quad + \frac{2}{k^2} \sum_{a \in K} \left[\sum_{b \in K} (f(abx) - f(bax)) \right] [f(x) - f(ax)] \\ &\quad + \frac{2}{k} \sum_{a \in K} \left\{ \frac{1}{k} \sum_{b \in K} [f(x) - f(bx)] - \frac{1}{k} \sum_{b \in K} [f(ax) - f(bax)] \right\} [f(x) - f(ax)] \\ &= -\frac{1}{k^2} \sum_{b \in K} \sum_{a \in K} [f(x) - f(ax) - f(bx) + f(abx)]^2 \\ &\quad + \frac{2}{k^2} \sum_{a \in K} \left[\sum_{b \in K} (f(abx) - f(bax)) \right] [f(x) - f(ax)] \\ &\quad + 2(\mathcal{L}f(x))^2 - \frac{2}{k} \sum_{a \in K} [f(x) - f(ax)] \mathcal{L}f(ax).\end{aligned}$$

On the other hand,

$$\begin{aligned}\mathcal{L}(f(x)\mathcal{L}f(x)) &= \frac{1}{k} \sum_{a \in K} [f(x)\mathcal{L}f(x) - f(ax)\mathcal{L}f(ax)] \\ &= \frac{1}{k} \sum_{a \in K} [f(x)\mathcal{L}f(x) - f(ax)\mathcal{L}f(x) + f(ax)\mathcal{L}f(x) - f(ax)\mathcal{L}f(ax)] \\ &= (\mathcal{L}f(x))^2 + \frac{1}{k} \sum_{a \in K} [f(ax) - f(x)] [\mathcal{L}f(x) - \mathcal{L}f(ax)] + \frac{1}{k} \sum_{a \in K} f(x) [\mathcal{L}f(x) - \mathcal{L}f(ax)] \\ &= (\mathcal{L}f(x))^2 + f(x)\mathcal{L}(\mathcal{L}f(x)) + \frac{1}{k} \sum_{a \in K} [f(ax) - f(x)] [\mathcal{L}f(x) - \mathcal{L}f(ax)].\end{aligned}$$

So, we get

$$\begin{aligned}\Gamma(f, \mathcal{L}f)(x) &= \frac{1}{2} \{f(x)\mathcal{L}(\mathcal{L}f(x)) + (\mathcal{L}f(x))^2 - \mathcal{L}(f(x)\mathcal{L}f(x))\} \\ &= \frac{1}{2k} \sum_{a \in K} [f(x) - f(ax)] [\mathcal{L}f(x) - \mathcal{L}f(ax)] \\ &= \frac{1}{2} (\mathcal{L}f(x))^2 - \frac{1}{2k} \sum_{a \in K} [f(x) - f(ax)] \mathcal{L}f(ax).\end{aligned}$$

Thus, we have

$$\begin{aligned}\Gamma_2(f, f)(x) &= \Gamma(f, \mathcal{L}f)(x) - \frac{1}{2} \mathcal{L}\Gamma(f, f)(x) \\ &= \frac{1}{2} (\mathcal{L}f(x))^2 - \frac{1}{2k} \sum_{a \in K} [f(x) - f(ax)] \mathcal{L}f(ax) \\ &\quad + \frac{1}{4k^2} \sum_{b \in K} \sum_{a \in K} [f(x) - f(ax) - f(bx) + f(abx)]^2 \\ &\quad - \frac{1}{2k^2} \sum_{a \in K} \left[\sum_{b \in K} (f(abx) - f(bax)) \right] [f(x) - f(ax)] \\ &\quad - \frac{1}{2} (\mathcal{L}f(x))^2 + \frac{1}{2k} \sum_{a \in K} [f(x) - f(ax)] \mathcal{L}f(ax) \\ &= \frac{1}{4k^2} \sum_{b \in K} \sum_{a \in K} [f(x) - f(ax) - f(bx) + f(abx)]^2 \\ &\quad - \frac{1}{2k^2} \sum_{a \in K} \left[\sum_{b \in K} (f(abx) - f(bax)) \right] [f(x) - f(ax)].\end{aligned}$$

This completes the proof. $\square$

Lemma 2.2. Suppose G is a homogeneous graph with edge generating set K consisting of k generators satisfying $CD(m, \xi)$, then for $x \in V$, we have

$$\begin{aligned}& -\frac{1}{k^2} \sum_{b \in K} \sum_{a \in K} [f(x) - f(ax) - f(bx) + f(abx)]^2 \\ & \quad + \frac{2}{k^2} \sum_{a \in K} \left[\sum_{b \in K} (f(abx) - f(bax)) \right] [f(x) - f(ax)] \\ & \leq -\frac{4}{m} (\mathcal{L}f(x))^2 - 2\xi\rho(x).\end{aligned}$$

Proof. Since G satisfies $CD(m, \xi)$, we have

$$\Gamma_2(f, f)(x) \geq \frac{1}{m} (\mathcal{L}f(x))^2 + \xi\Gamma(f, f)(x).$$

By Lemma 2.1 and the above inequality, we get

$$\begin{aligned} & \frac{1}{4k^2} \sum_{b \in K} \sum_{a \in K} [f(x) - f(ax) - f(bx) + f(abx)]^2 \\ & \quad - \frac{1}{2k^2} \sum_{a \in K} \left[\sum_{b \in K} (f(abx) - f(bax)) \right] [f(x) - f(ax)] \\ & \geq \frac{1}{m} (\mathcal{L}f(x))^2 + \frac{\xi}{2} \rho(x). \end{aligned}$$

Arranging the above inequality, we get the result as stated in this lemma. $\square$

By Lemma 2.2, we can generate a very similar Harnack inequality as in [3] in the following Theorem 2.3. The proof is similar, so we omit the proof here.

Theorem 2.3. Suppose G is a homogeneous graph satisfying the curvature-dimension type inequality $CD(m, \xi)$, and $f \in V^R$ is a harmonic eigenfunction of Laplacian $\mathcal{L}$ with eigenvalue $\lambda \neq 0$, then the following inequality holds for all $x \in V$

$$\frac{1}{k} \sum_{a \in K} [f(x) - f(ax)]^2 \leq \left[\left(8 - \frac{2}{m} \right) \lambda - 4\xi \right] \max_{z \in V} f^2(z).$$

Using the Harnack inequality in Theorem 2.3, we can derive a lower eigenvalue estimate as follows, which is very similar to the one in [3], and is useful for us to get a lower bound for the log-Sobolev constant in Theorem 1.4.

Theorem 2.4. Suppose G is a homogeneous graph satisfying $CD(m, \xi)$, and λ is a non-zero eigenvalue of Laplace operator $\mathcal{L}$ on G . Then

$$\lambda \geq \frac{1 + 4\xi k D^2}{k(8 - \frac{2}{m})D^2},$$

where D is the diameter of G .

3. The proof of Theorem 1.3

In this section, by Lemma 2.2 and following the proof of Theorem 6 in [5], we give the proof of Theorem 1.3.

Proof of Theorem 1.3. We set

$$\rho(x) = \frac{1}{k} \sum_{a \in K} [f(x) - f(ax)]^2.$$

Then, by Lemma 2.2, we have

$$\begin{aligned} \mathcal{L}\rho(x) &= \frac{1}{k^2} \sum_{b \in K} \sum_{a \in K} \{ [f(x) - f(ax)]^2 - [f(bx) - f(abx)]^2 \} \\ &= -\frac{1}{k^2} \sum_{b \in K} \sum_{a \in K} [f(x) - f(ax) - f(bx) + f(abx)]^2 \end{aligned}$$

$$\begin{aligned}
& + \frac{2}{k^2} \sum_{a \in K} \left[\sum_{b \in K} (f(abx) - f(bax)) \right] [f(x) - f(ax)] \\
& + \frac{2}{k^2} \sum_{b \in K} \sum_{a \in K} [f(x) - f(ax) - f(bx) + f(bax)] [f(x) - f(ax)] \\
& \leq -\frac{4}{m} (\mathcal{L}f(x))^2 - 2\xi\rho(x) + X,
\end{aligned} \tag{3.1}$$

where X denotes the third term in the second equality, and by [Lemma 1.2](#), we get

$$\begin{aligned}
X &= \frac{2}{k} \sum_{a \in K} \left\{ \frac{1}{k} \sum_{b \in K} [f(x) - f(bx)] - \frac{1}{k} \sum_{b \in K} [f(ax) - f(bax)] \right\} [f(x) - f(ax)] \\
&= \frac{2\beta}{k} \sum_{a \in K} [f(x) \log f^2(x) - f(ax) \log f^2(ax)] [f(x) - f(ax)].
\end{aligned} \tag{3.2}$$

Just almost the same as the proof of Theorem 6 in [\[5\]](#), for any $\sigma \geq 2$, we get

$$\begin{aligned}
\mathcal{L}(\sigma\beta f^2(x) \log f^2(x)) + X &= \frac{\sigma\beta}{k} \sum_{a \in K} [f^2(x) \log f^2(x) - f^2(ax) \log f^2(ax)] + X \\
&= 2\sigma\beta^2 f^2(x) \log^2 f^2(x) \\
&\quad - \frac{(\sigma-2)\beta}{k} \sum_{a \in K} [f(x) \log f^2(x) - f(ax) \log f^2(ax)] [f(x) - f(ax)] \\
&\quad + \frac{\sigma\beta}{k} \sum_{a \in K} f(x)f(ax) [\log f^2(x) - \log f^2(ax)] \\
&\leq 2\sigma\beta^2 f^2(x) \log f^2(x) (1 + \log f^2(x)) \\
&\quad - \frac{(\sigma-2)\beta}{k} \sum_{a \in K} [f(x) \log f^2(x) - f(ax) \log f^2(ax)] [f(x) - f(ax)] \\
&\leq 2\sigma\beta^2 f^2(x) \log f^2(x) (1 + \log f^2(x)) \\
&\quad - (\sigma-2)\beta^4 f^2(x) |\log^3 f^2(x)| - 2(\sigma-2)\beta\rho(x) \\
&\leq 2\sigma\beta^2 f^2(x) \log f^2(x) (1 + \log f^2(x)) - 2(\sigma-2)\beta\rho(x).
\end{aligned} \tag{3.3}$$

Thus, by [\(3.1\)](#), [\(3.2\)](#), [\(3.3\)](#) and [Lemma 1.2](#), we have

$$\begin{aligned}
\mathcal{L}(\rho(x) + \sigma\beta f^2(x) \log f^2(x)) &\leq -\frac{4}{m} (\mathcal{L}f(x))^2 - 2\xi\rho(x) + \mathcal{L}(\sigma\beta f^2(x) \log f^2(x)) + X \\
&\leq 2\sigma\beta^2 f^2(x) \log f^2(x) (1 + \log f^2(x)) - \frac{4}{m} \beta^2 f^2(x) \log^2 f^2(x) \\
&\quad - 2(\sigma\beta - 2\beta + \xi)\rho(x).
\end{aligned}$$

Now we consider a vertex v which achieves the maximum value for $\rho(x) + \sigma\beta f^2(x) \log f^2(x)$ over all $x \in V$, and we have

$$\begin{aligned}
0 &\leq \mathcal{L}(\rho(v) + \sigma\beta f^2(v) \log f^2(v)) \\
&\leq 2\sigma\beta^2 f^2(v) \log f^2(v) (1 + \log f^2(v)) - \frac{4}{m} \beta^2 f^2(v) \log^2 f^2(v) - 2(\sigma\beta - 2\beta + \xi)\rho(v).
\end{aligned}$$

Taking $\xi = s\beta$ and $s > 0$, we get

$$\rho(v) \leq \frac{2\sigma\beta f^2(v) \log f^2(v)(1 + \log f^2(v)) - \frac{4}{m}\beta f^2(v) \log^2 f^2(v)}{2(\sigma - 2 + s)}.$$

Therefore for every $x \in V$ and any $\sigma \geq 2$, we have

$$\begin{aligned} & \frac{1}{k} \sum_{a \in K} [f(x) - f(ax)]^2 + \sigma\beta f^2(x) \log f^2(x) \\ & \leq \rho(v) + \sigma\beta f^2(v) \log f^2(v) \\ & \leq \left(\frac{\sigma\beta}{\sigma - 2 + s} + \sigma\beta \right) \cdot f^2(v) \log f^2(v)(1 + \log f^2(v)) - \frac{2\beta}{m(\sigma - 2 + s)} \cdot f^2(v) \log^2 f^2(v) \\ & \leq \frac{\sigma\beta(\sigma - 1 + s)}{\sigma - 2 + s} \cdot f^2(v) \log f^2(v)(1 + \log f^2(v)) - \frac{2\beta}{m(\sigma - 2 + s)} \cdot f^2(v) \log^2 f^2(v) \\ & = \frac{\sigma\beta(\sigma - 1 + s)}{\sigma - 2 + s} \cdot f^2(v) \log f^2(v) \cdot \left[1 + \left(1 - \frac{2}{m\sigma(\sigma - 1 + s)} \right) \log f^2(v) \right]. \end{aligned}$$

We consider the following cases:

1. When $|f(x)| \geq 1$, we can drop the term $\sigma\beta f^2(x) \log f^2(x)$, and get

$$\sum_{a \in K} [f(x) - f(ax)]^2 \leq \frac{k\sigma\beta(\sigma - 1 + s)}{\sigma - 2 + s} \cdot f^2(v) \log f^2(v) \cdot \left[1 + \left(1 - \frac{2}{m\sigma(\sigma - 1 + s)} \right) \log f^2(v) \right].$$

2. When $|f(x)| < 1$ and $f(x) \neq 0$, we can check $-f^2(x) \log f^2(x) \leq e^{-1} < 1$, and so we have

$$\sum_{a \in K} [f(x) - f(ax)]^2 \leq \frac{k\sigma\beta(\sigma - 1 + s)}{\sigma - 2 + s} f^2(v) \log f^2(v) \left[1 + \left(1 - \frac{2}{m\sigma(\sigma - 1 + s)} \right) \log f^2(v) \right] + k\sigma\beta.$$

Since $\sigma \geq 2$, $m \geq 1$, $s > 0$, we have $0 < \frac{2}{m\sigma(\sigma - 1 + s)} < 1$. Thus, for $U = \sup_x |f(x)| \geq 1$, we get

$$\sum_{a \in K} [f(x) - f(ax)]^2 \leq \frac{2k\sigma\beta(\sigma - 1 + s)}{\sigma - 2 + s} \cdot \max \left\{ U^2 \log U^2 \left[1 + \left(1 - \frac{2}{m\sigma(\sigma - 1 + s)} \right) \log U^2 \right], 1 \right\}.$$

This completes the proof. $\square$

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